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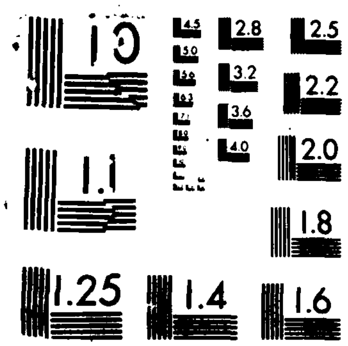
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# 2-EXTENDABILITY IN 3-POLYTOPES

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## 1. Introduction and Terminology

Suppose $G$ is a graph with $p$ points and let $n$ be a positive integer such that $p \geq 2n+2$. Graph $G$ is said to be **n-extendable** if every matching of size $n$ in $G$ extends to a perfect matching. (We ~~will abbreviate~~ the term "perfect matching" to "~~p.m.~~" ~~hereafter.~~) A graph $G$ is called **bicritical** if $G-u-v$ has a perfect matching, for all pairs of points $u, v \in V(G)$. In [~~LP2~~] a canonical decomposition theory for graphs in terms of their maximum (or, when present, perfect) matchings is discussed at length. Bicritical graphs play an important roll in this theory. In particular, those bicritical graphs which are 3-connected (the so-called **bricks**) currently represent the "atoms" of the decomposition theory in that no further decomposition of these graphs has been obtained as yet. Indeed at present we seem far from an understanding of the structure of bicritical graphs or even that of bricks.

Although interesting in its own right, the study of $n$-extendability became more important when in [Plu1] it was shown that every 2-extendable non-bipartite graph is a brick and that, for $n \geq 2$, every $n$-extendable graph is also $(n-1)$-extendable. Thus we have available for study a nested set of subcollections of bicritical graphs.

The results of the present paper will be presented in two parts. In Section 2, we present some new procedures for constructing infinite

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families of bricks by means of constructing families of non-bipartite 2-extendable graphs.

In Section 3, we focus our attention on extending matchings in planar graphs. Planar 1-extendable graphs abound; for example, every cubic 3-polytopal graph has this property. On the other hand, it has been shown [Plu2] that no planar graph is 3-extendable. Between these extremes, then, lies the class of planar 2-extendable graphs.

We will further restruct our attention in Section 3 to cubic 3-connected planar graphs (or **cubic polytopal** graphs, as they are sometimes called). Which of these graphs are 2-extendable? Our main result states that any cubic polytopal graph which is cyclically 4-connected, but has no quadrilateral face, is 2-extendable. (Clearly, if such a graph has no quadrilateral face, it can have no cycle of length 4 at all.) In particular, cubic polytopal graphs with cyclic connectivity at least 5 must be 2-extendable.

## 2. Some families of bicritical and 2-extendable graphs

The complete graphs on an even number of points, $\{K_{2r}\}_{r=1}^{\infty}$ are trivially bicritical, as are the *wheel graphs* with even total number of points. (A **wheel graph** is obtained by joining every point of a cycle to a common point (or "hub") not on the cycle.) The first non-trivial class of graphs proven to be bicritical seems to be the class of **Halin** graphs. (See [LP1]. These graphs have also been called **based polyhedra** by Rademacher [R1] and **roofless polyhedra** by Pólya.) These graphs are sometimes indicated by $T \cup C$ where $T$ is a tree on an even number of points in which each non-endpoint has minimum degree 3, and $C$ is a cycle through the endpoints of tree $T$ so that $T \cup C$ is planar.

For $i = 1, 2$, let $G_i$ be a graph containing a point $v_i$ of degree 3. Further suppose that the neighbors of $v_i$ in $G_i$ are $\{x_i, y_i, z_i\}$. Let us denote by $G_1(v_1v_2)G_2$ (or simply $G_{1,2}$ when the $v_i$'s are understood) the graph obtained from $G_1$ and $G_2$ by deleting points $v_1$ and $v_2$ and then inserting the lines $x_1x_2, y_1y_2$ and $z_1z_2$. We shall have occasion to call this operation **3-joining**. (See Figure 2.1.)

**2.1. THEOREM.** *Suppose $G_1$, $G_2$, $v_1$, $v_2$ and $G_{1,2}$ are as given above. Then:*

(*a*) *if $G_1$ and $G_2$ are bicritical, so is $G_{1,2}$.*

(*b*) *if $G_1$ and $G_2$ are 2-extendable and non-bipartite, so is $G_{1,2}$.*

**PROOF.** Let $G'_i = G_i - v_i$, for $i = 1, 2$. To prove (a), let $u_1$ and $u_2$ be two distinct points in graph $G_{1,2}$. First suppose $u_1, u_2 \in V(G'_1)$. Let

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**FIGURE 2.1.**

$P_1$ be a p.m. of $G_1 - u_1 - u_2$ and without loss of generality let $v_1x_1$ be the line of $P_1$ covering point $v_1$. Then let $P_2$ be a p.m. of $G_2$ containing line $v_2x_2$. (Remember that every bicritical graph is 1-extendable.) Then $P_1 \cup P_2 - v_1x_1 - v_2x_2 + x_1x_2$ is a p.m. for $G_{1,2} - u_1 - u_2$.

Now suppose $u_1 \in V(G'_1)$ and $u_2 \in V(G'_2)$. Let $P_i$ be a p.m. for $G_i - u_i - v_i$ for $i = 1, 2$. Then $P_1 \cup P_2$ is a p.m. in $G_{1,2}$.

To prove (b), let $e_1$ and $e_2$ be two independent lines in $G_{1,2}$.

Before proceeding, let us note that since each $G_i$ is 2-extendable, $|V(G_i)| \geq 6$ and hence $\{x_i, y_i, z_i\}$ is a cutset in $G_i$. Moreover, by Theorem 3.2 of [Plu1], each $G_i$ is 3-connected. But then by Theorem 2.2 of [Plu3] each cutset $\{x_i, y_i, z_i\}$ is independent in $G_i$.

Now suppose that $\{e_1, e_2\} \subseteq V(G'_1)$. Let $P_1$ be a p.m. of $G_1$ containing $e_1$ and $e_2$. Since $\{x_1, y_1, z_1\}$ is independent, without loss of generality we may assume that $x_1v_1$ is the line of $P_1$ covering point $v_1$. Then if $P_2$ is a p.m. of $G_2$ containing line $x_2v_2$, the matching $P_1 \cup P_2 - x_1v_1 - x_2v_2 + x_1x_2$ is a p.m. for $G_{1,2}$ containing $e_1$ and $e_2$.

Secondly, suppose $e_1 \in E(G'_1)$ and $e_2 \in E(G'_2)$. Now line $e_i$ meets at most one of the points $x_i, y_i$ and $z_i$, since $\{x_i, y_i, z_i\}$ is independent in $G_i$, so among the pairs $\{x_1, x_2\}$, $\{y_1, y_2\}$ and $\{z_1, z_2\}$ we can choose one - say $\{x_1, x_2\}$ - such that neither $x_1$ nor $x_2$ is covered by either $e_1$ or $e_2$. For $i = 1, 2$, let $P_i$ be a p.m. for $G_i$ containing $e_i$ and $x_iv_i$. Then $P_1 \cup P_2 - x_1v_1 - x_2v_2 + x_1x_2$ is a p.m. for $G_{1,2}$ containing $e_1$ and $e_2$.

Thirdly, suppose $e_1 = x_1x_2$ and $e_2 \in E(G'_2)$. Let $P_1$ be a p.m. for $G_1$ containing $x_1v_1$ and let $P_2$ be a p.m. for $G_2$ containing $x_2v_2$ and $e_2$. Then $P_1 \cup P_2 - x_1v_1 - x_2v_2 + x_1x_2$ is a p.m. for $G_{1,2}$ containing $e_1$ and $e_2$.

Finally, suppose $\{e_1, e_2\} \subseteq \{x_1x_2, y_1y_2, z_1z_2\}$, say without loss of generality that $e_1 = x_1x_2$ and $e_2 = y_1y_2$. Now since each $G_i$ is non-bipartite, it is bicritical by Theorem 4.2 of [Plu1] and hence $G_i - x_i - y_i$ has a p.m. $P_i$. Moreover, $P_i$ must contain line $v_i z_i$. But then $P_1 \cup P_2 - v_1 z_1 - v_2 z_2 + z_1 z_2 + e_1 + e_2$ is a p.m. for $G_{1,2}$ containing lines $e_1$ and $e_2$.

It only remains to show that $G_{1,2}$ is non-bipartite. But since $G_1$ and $G_2$ are both non-bipartite, they are bicritical (again using Theorem 4.2 of [Plu1]) and hence by part (a) of the present theorem, $G_{1,2}$ is also bicritical. But no bicritical graph is bipartite and this completes the proof. ∎

Since the operation of 3-joining preserves the properties of being cubic, 3-connected and planar, it may be used to obtain infinite families of bicritical polytopal graphs and infinite families of 2-extendable cubic 3-polytopal graphs. For example, let $G_1$ and $G_2$ be two copies of the dodecahedron and let $v_i$ be any point in $G_i$ for $i = 1, 2$. Then $G_{1,2}$ is again 2-extendable and, since it is non-bipartite, it is also bicritical.

Let us point out that the complete graph on four points, $K_4$, is a much smaller starting graph for generating bicritical graphs by 3-joining than is the dodecahedron, but $K_4$ is not 2-extendable. Note that if one joins two copies of $K_4$ together via a 3-joining, the resulting 6-point graph (the so-called *triangular pyramid*) is not 2-extendable either.

It is also interesting to point out that, although 3-joining preserves 2-extendability in non-bipartite graphs, this operation when applied to bipartite graphs, *never* preserves 2-extendability! More particularly, see Corollary 2.3 which follows the next theorem.

A set of lines $L = \{e_1, \ldots, e_k\} \subseteq E(G)$ is a **cyclic $k$-cut** in a connected graph $G$ if $G - L$ consists of two components each of which contains a cycle.

**2.2. THEOREM.** *Suppose $G$ is a 2-extendable graph with a cyclic 3-cut $L = \{e_1, e_2, e_3\}$. Then if $H_1$ and $H_2$ are the components of $G - L$, neither $H_1$ nor $H_2$ is bipartite.*

**PROOF.** Suppose $L = \{e_1, e_2, e_3\}$ is a cyclic 3-cut and that $e_1 = x_1 x_2$, $e_2 = y_1 y_2$ and $e_3 = z_1 z_2$ where $\{x_i, y_i, z_i\} \subseteq V(H_i)$. Let $H'_i = H_i - x_i - y_i - z_i$ for $i = 1, 2$.

First we claim that $L$ is a matching. For suppose not. Without loss of generality, we may suppose that $e_1$ and $e_2$ are adjacent at point $x_1 = y_1$. Then since $H_1$ contains a cycle, it must contain a point $u \notin \{x_1, y_1, z_1\}$. But then $\{x_1, z_1\}$ is a cutset in $G$. (We point out that $z_1$ may or may not be different from $x_1 = y_1$ here.) But this contradicts the fact that $G$ must be 3-connected by Theorem 3.2 of [Plu1] and the claim is proved.

Suppose that $V(H'_1) = \emptyset$. Then since $G$ is 3-connected, $H_1$ must be a triangle on points $x_1, y_1$ and $z_1$. But then let $f$ be any line in $H_2$ incident with $x_2, y_2$ or $z_2$. Without loss of generality, suppose $f$ is incident with $x_2$. Then lines $\{f, y_1z_1\}$ do not extend to a p.m. in $G$, a contradiction.

Thus we may assume both $V(H'_1) \neq \emptyset$ and $V(H'_2) \neq \emptyset$. However, then for both $i = 1, 2$, the set $\{x_i, y_i, z_i\}$ is a point cutset in $G$ and again by Theorem 2.2 of [Plu3], each is independent. But then since $\operatorname{mindeg} G \geq 3$, it follows immediately that $|V(H'_i)| \geq 2$, for $i = 1, 2$.

Suppose now that $H_2$ is bipartite. There are then two cases to treat, depending upon the parity of $|V(H_2)|$.

(a) Suppose first that $|V(H_2)|$ is even. Let the bipartition of $H_2$ be $(A_2, B_2)$. Then among the points $x_2, y_2, z_2$ some two must belong to the same color class. Without loss of generality, suppose that $x_2, y_2 \in A_2$. Now let $P_1$ be a p.m. of $G$ containing lines $e_1$ and $e_2$. By parity, $P_1$ must match $z_2$ into $H'_2$. Hence $|A_2| = |B_2| + 2$.

But now since $|V(H'_2)| \geq 2$, and $\{x_2, y_2, z_2\}$ is an independent cutset, there must be two independent lines joining $x_2$ and $y_2$ to two points in $V(H'_2)$. Call these lines $f_1$ and $f_2$. Let $P_2$ be a p.m. of $G$ containing lines $f_1$ and $f_2$. Then by parity, since $H_2$ is even, $P_2$ must match $z_2$ into $H'_2$ as well. So $P_2$ restricted to $H_2$ is a p.m. of $H_2$ and hence $|A_2| = |B_2|$, a contradiction. Hence $|V(H_2)|$ cannot be even.

Suppose, therefore, that $|V(H_2)|$ is odd. As before, we may assume that $(A_2, B_2)$ is the bipartition of $H_2$ with $x_2, y_2 \in A_2$.

First suppose that $z_2 \in A_2$ as well. Let $P_3$ be a p.m. for $G$ containing $e_1$ and $e_2$ — and hence by parity — $e_3$ as well. Thus $|A_2| = |B_2| + 3$. Once again, as in Case (a) above, there must be two independent lines $f_1$ and $f_2$ matching $x_2$ and $y_2$ into $H'_2$. Let $P_4$ be a p.m. of $G$ containing $f_1$ and $f_2$. Then by parity, $P_4$ contains $e_3$. But then $|A_2| = |B_2| + 1$, a contradiction.

Thus we may assume that $z_2 \in B_2$. Let $P_5$ be a p.m. of $G$ containing $e_1$ and $e_2$ and — by parity — $e_3$ as well. Thus $|A_2| = |B_2| + 1$. But as before, without loss of generality, we may assume there are two independent lines $f_1$ and $f_2$ matching $x_2$ and $y_2$ into $H'_2$. Let $P_6$ be a p.m. for $G$ containing lines $f_1$ and $f_2$. Then, by parity, line $e_3 = z_1z_2 \in P_6$. Then $|A_2| = |B_2| - 1$, a contradiction.

Thus $H_2$ is not bipartite and, similarly, neither is $H_1$. ∎

**2.3. COROLLARY.** *If $G$ is a 2-extendable bipartite graph, then $G$ is cyclically 4-connected.* ∎

We can in fact be a bit more precise about just which sets of two independent lines extend to p.m.'s in a graph obtained from two bipartite 2-extendable graphs by 3-joining. To this end we provide the following result.

**2.4. THEOREM.** *Let $G_1$ and $G_2$ be 2-extendable bipartite graphs and suppose $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ are both points of degree 3. Then:*
*(a) the 3-join graph $G_1(v_1v_2)G_2 = G_{1,2}$ is 1-extendable and bipartite and*
*(b) if $e_1$ and $e_2$ are two independent lines in $G_{1,2}$ then they extend to a p.m. of $G_{1,2}$ if and only if at least one of $e_1$ and $e_2$ is not a join line.*

**PROOF.** Suppose the bipartition of $G_i$ is $A_i \cup B_i$ where $v_i \in A_i$ for $i = 1,2$. Then $|A_i| = |B_i|$ and $(B_1 \cup A_2 - v_2) \cup (A_1 \cup B_2 - v_1)$ is a bipartition of $G_{1,2}$.

Now let $e_1$ and $e_2$ be independent lines in $G_{1,2}$. If $\{e_1, e_2\} \subseteq E(G_1')$, if $e_1 \in E(G_1')$ and $e_2 \in E(G_2')$, or if $e_1$ is a join line and $e_2 \in E(G_2')$, say, then $\{e_1, e_2\}$ extends to a p.m. of $G_{1,2}$ by Theorem 2.1(b).

So suppose $e_1 = x_1x_2$ and $e_2 = y_1y_2$ are both join lines. Further, suppose that $P$ is a p.m. for $G_{1,2}$ containing $e_1$ and $e_2$. Then since $|V(G_i')|$ is odd for $i = 1,2$, $P$ must contain $e_3 = z_1z_2$ as well. But $\{x_1, y_1, z_1\} \subseteq B_1$ and hence $P$ must match $B_1' = B_1 - \{x_1, y_1, z_1\}$ onto $A_1 - v_1$ which is impossible since $|B_1'| = |A_1 - v_1| - 2$. Thus there is no p.m. for $G_{1,2}$ containing $e_1$ and $e_2$. This completes the proof of part (b).

It remains only to show that every join line extends to a p.m. of $G_{1,2}$. Let us consider the join line $e = x_1x_2$. Now each $G_i$ is 1-extendable by Theorem 2.2 of [Plu1]. Thus let $P_i$ be a p.m. for $G_i - x_i - v_i$ for $i = 1,2$. Then $P = P_1 \cup P_2 - x_1v_1 - x_2v_2 + x_1x_2$ is a p.m. for $G_{1,2}$. ■

We now present a second construction. Again for $i = 1,2$, let $G_i$ be a graph containing a point $v_i$ of degree 3. Suppose once again that the neighbors of $v_i$ in $G_i$ are $\{x_i, y_i, z_i\}$. Let us denote by $G_1\text{hex}(v_1, v_2)G_2$ (or simply $G_1\text{hex}G_2$ when the $v_i$'s are understood) the graph obtained from $G_1$ and $G_2$ by deleting $v_1$ and $v_2$, adding a hexagon on 6 new points, $a_1a_2b_1b_2c_1c_2a_1$, and adding lines $x_ia_i$, $y_ib_i$ and $z_ic_i$ for $i = 1,2$. We will call $G_1\text{hex}G_2$ a **hex-join** of $G_1$ and $G_2$. (See Figure 2.2.)

We then have the following result which is parallel to Theorem 2.1.

**2.5. THEOREM.** *Suppose $G_1, G_2, v_1, v_2$ and $G_1\text{hex}G_2$ are as given above. Then:*
*(a) If $G_1$ and $G_2$ are bicritical, so is $G_1\text{hex}G_2$, and*

FIGURE 2.2.

(b) *If* $G_1$ *and* $G_2$ *are 2-extendable and non-bipartite, then so is* $G_1\text{hex}G_2$.

**PROOF.** (a). The proof here is in much the same spirit as the proof of part (a) of Theorem 2.1. Hence we will treat only one representative case and leave the rest to the reader. (Again we adopt the labeling shown in Figure 2.2. The reader should observe that the symmetry displayed in Figure 2.2 substantially reduces the number of cases which need to be treated.)

Let us suppose that $u \in \{a_1, b_1, c_1\}$. Without loss of generality, suppose that $u_1 = a_1$. Also suppose $v \in \{a_2, b_2, c_2\}$. By symmetry we need treat only two subcases, namely when $v = a_2$ and when $v = b_2$.

If $v = a_2$, let $P_1$ be a p.m. of $G - y_1 - v_1$ and let $P_2$ be a p.m. of $G_2 - y_2 - v_2$. Then $P_1 \cup P_2 + y_1b_1 + y_2b_2 + z_1c_2$ is a p.m. for $(G_1\text{hex}G_2) - u - v$. If $v = b_2$, let $P_1$ be a p.m. for $G_1 - y_1 - v_1$ and $P_2$, a p.m. for $G_2 - x_2 - v_2$. Then $P_1 \cup P_2 + y_1b_1 + x_2a_2 + c_1c_2$ is a p.m. for $(G_1\text{hex}G_2) - u - v$.

Similarly, the proof of part (b) here mimics that of Theorem 2.1. We therefore present only two representative cases, one in which the bicriticality of $G_1$ and $G_2$ is used (and hence the assumption that each is non-bipartite) and one in which it is not used.

So first let $e_1 = x_1a_1$ andd $e_2 = y_1b_1$. We seek a p.m. for

$G_1\text{hex}G_2$ containing $e_1$ and $e_2$. Since $G_1$ is bicritical, there is a p.m. $P_1$ for $G_1 - x_1 - y_1$ and it must contain line $z_1v_1$. Similarly, there is a p.m. $P_2$ for $G_2 - x_2 - y_2$ and it must contain line $z_2v_2$. But then $P_1 \cup P_2 - z_1v_1 - z_2v_2 + x_1a_1 + y_1b_1 + z_1c_1 + x_2a_2 + y_2b_2 + z_2c_2$ is a p.m. for $G_1\text{hex}G_2$ containing $e_1$ and $e_2$.

Finally, let $e_1 = a_1a_2$ and $e_2 = b_1b_2$. Then let $P_1$ be a p.m. for $G_1$ containing $z_1v_1$ and $P_2$ be a p.m. for $G_2$ containing $z_2v_2$. Then $P_1 \cup P_2 - z_1v_1 - z_2v_2 + z_1c_1 + z_2c_2 + a_1a_2 + b_1b_2$ is a p.m. for $G_1\text{hex}G_2$ containing $e_1$ and $e_2$. ∎

## 3. The main result

Recall from the previous section that the construction procedures called 3-joining and hex-joining preserve the properties of 3-regularity, 3-connectivity, planarity, bicriticality and 2-extendability. On the other hand, since each of these operations automatically inserts a cyclic cutset of size 3, cyclic connectivity is *not* necessarily preserved.

A question which arose early in the studies culminating in this paper was whether or not a cubic 3-connected planar graph (hereafter called a **simple 3-polytopal graph**) of sufficiently high cyclic connectivity must be 2-extendable. (For more information on polytopal graphs, the reader is referred to the classical book of Grünbaum [G1]. Suffice it to say, for our purposes, that the 3-connected planar graphs are called polytopal because they are just the skeleta of 3-polytopes by a celebrated theorem of Steinitz [S1].)

Examples showing that cyclic 3- and 4-connectivity are not sufficient to insure 2-extendability in cubic 3-polytopal graphs are presented at the end of this section.

We now present our main result.

**3.1. THEOREM.** *If $G$ is a cubic 3-polytopal graph which is cyclically 4-connected and has no faces of size 4, then $G$ is 2-extendable.*

**PROOF.** Before proceeding, we would point out that the hypotheses of this theorem imply that $G$ cannot have any 3-cycles or 4-cycles.

Now suppose $G$ satisfies the hypotheses of the theorem, but is not 2-extendable. So let $e_1 = x_1y_1$ and $e_2 = x_2y_2$ be two independent lines in $G$ which do not lie in a p.m. for $G$. Thus graph $G' = G - x_1 - y_1 - x_2 - y_2$ has no perfect matching and hence, by Tutte's classical theorem on perfect matchings, there is a set $S' \subseteq V(G)$ such that $c_o(G' - S') > |S'|$, where $c_o(G' - S')$ denotes the number of odd components of $G' - S'$. Then since $|V(G')|$ is even, parity dictates that $c_o(G' - S') \geq |S'| + 2$.

Suppose in fact that $c_o(G'-S') \geq |S'|+3$ and hence again by parity that $c_o(G'-S') \geq |S'|+4$. Now $G$ is 1-extendable, by a result of Plesník [Ple1] (and independently by a result of Little, Grant and Holton [LGH1, LGH2]). So line $e_2 = x_2y_2$ lies in a p.m. for $G$ and hence $G'' = G-x_2-y_2$ has a p.m. But then in $G''$ we have a set $S'' = S' \cup \{x_1, y_1\}$ such that $c_o(G'' - S'') = c_o(G' - S') \geq |S'| + 4 = |S''| + 2$. But then by Tutte's Theorem, graph $G''$ has no p.m., a contradiction. Thus $c_o(G' - S') = |S'| + 2$.

Let $S = S' \cup \{x_1, y_1, x_2, y_2\}$. We claim that $G - S$ has no even components. For suppose $C_e$ were such an even component. Then since $G$ is 3-connected, there must be at least 3 (and hence by parity, at least 4) lines from $C_e$ to $S$. These lines, together with $e_1$ and $e_2$, imply that no more than $3|S|-8$ lines are sent from $S$ to the odd components of $G-S$. But viewing these lines from $G - S$, each odd component must send at least 3 lines to $S$ and hence there are at least $3(|S|-2) = 3|S|-6$ of these lines, so we have a contradiction. Thus $G - S$ has no even components.

Let $N$ denote the number of lines joining $S$ to $G - S$. Note that since $G$ is 3-connected, each odd component of $G-S$ must send at least 3 lines to $S$ and hence $N \geq 3(|S|-2) = 3|S|-6$. So we have the inequality $3|S| - 6 \leq N \leq 3|S| - 4$. Accordingly, there are three cases to consider.

**Case 1.** Suppose $N = 3|S| - 6$. So in $S$ there exists precisely one more line $e_3$, in addition to lines $e_1$ and $e_2$, and each odd component of $G - S$ sends exactly 3 lines up to $S$. Thus up to a relabeling of the three $e_i$, $G$ has the appearance of one of the three graphs shown in Figure 3.1.

Henceforth we will denote by $C_1, C_2, \ldots, C_{|S|-2}$ the odd components of $G - S$.

Suppose now that all the $C_i$'s are singletons. Then a well-known variation on Euler's formula relating the number of points, lines and faces of any planar graph yields $\sum_i (6-i)f_i = 12$, where $f_i$ denotes the number of faces containing $i$ lines in their boundary. Hence $3f_3 + 2f_4 + f_5 \geq 12$ and since $f_3 = f_4 = 0$, we must have $f_5 \geq 12$. But $G - e_1 - e_2 - e_3$ is bipartite, and it then follows that $f_5 \leq 6$, a contradiction.

Hence we may assume that there exists one of the $C_i$ — say $C_1$ — with $|V(C_1)| \geq 3$.

**Claim 1.** Component $C_1$ contains a cycle.

For suppose not. Then it must be a tree with at least 2 endpoints and hence it must send at least 4 lines to $S$, a contradiction.

**Claim 2.** Subgraph $G_1 = G[V(G) - V(C_1)]$ contains a cycle.

FIGURE 3.1.

Suppose not. Then $G_1$ is a forest containing at least 3 lines, so it must contain at least 2 endpoints. But then $G_1$ must send at least 4 lines down to $C_1$, again a contradiction.

So we have shown that the 3 lines joining $C_1$ to $G_1$ are a *cyclic* cutset of size 3, contradicting the hypothesis that $G$ is cyclically 4-connected.

**Case 2.** Suppose $3|S|-5$. But then since there are exactly $|S|-2$ odd components in $G-S$ and each must send at least 3 lines to $S$ by the 3-connectivity of $G$, we must have one odd component of $G-S$ sending exactly 4 lines to $S$ and all the rest sending exactly 3. But $G$ is cubic, so it is impossible for any odd component of $G-S$ to send an even number of lines to $S$ and we have a contradiction.

**Case 3.** So we may assume that $3|S|-4=N$. So there must be exactly two lines in the induced subgraph $G[S]$ and they must be $e_1$ and $e_2$. Since no odd component of $G-S$ can send exactly 4 lines to $S$ by parity, but all odd components must send at least 3 lines to $S$ by 3-connectivity, we must have exactly one odd component of $G-S$, without loss of generality say it is $C_1$, sending at least 5 lines to $S$. (So component $C_1$ must contain at least 3 points.) But then we have $3|S|-4=N\geq 5+3(|S|-3)=3|S|-4$, and it follows that we must have *exactly one* odd component which sends *exactly* 5 lines to $S$, this odd

FIGURE 3.2.

component must contain at least 3 points and all other odd components of $G - S$ send exactly 3 lines to $S$.

Let $C_2$ be any other odd component of $G - S$ different from $C_1$. Suppose component $C_2$ is not a singleton.

By the arguments of Claims 1 and 2 of Case 1 above, component $C_2$ and subgraph $G[V(G) - V(C_2)]$ both contain cycles.

But then we have a cyclic cutset joining $C_2$ and $G_2$ containing exactly 3 lines, contradicting the hypothesis that $G$ is cyclically 4-connected.

Thus component $C_1$ is an odd component sending exactly 5 lines to $S$ and all the remaining odd components of $G - S$, namely, $C_2, \dots, C_{|S|-2}$, are *singletons* incident with exactly 3 lines from $S$. (See Figure 3.2.)

**Claim 3.** Component $C_1$ contains at least 5 points.

Suppose not. Then $|V(C_1)| = 3$. But then $C_1$ must be a path of length 2, since we know graph $G$ contains no triangles. Let the two adjacent lines of $C_1$ be denoted by $e_5$ and $e_6$. As before, using the Euler formula to do a face count, we have $3f_3 + 2f_4 + f_5 \geq 12$ and, since $f_3 = f_4 = 0$, we have $f_5 \geq 12$. But $G - e_1 - e_2 - e_5 - e_6$ is bipartite and hence $G$ has $f_5 \leq 8$, a contradiction and Claim 3 is proved.

**Claim 4.** Component $C_1$ contains a cycle.

If not, it must be a tree with at least 2 endpoints and hence sends at least 4 lines to $S$. If it had at least 3 endpoints, it would have to send at least 6 lines to $S$, a contradiction. Thus tree $C_1$ contains exactly 2 endpoints and hence must be a path. But since $C_1$ sends exactly 5 lines to $S$, it must be a path of length 2, contradicting Claim 3.

Let $G'$ denote the graph obtained from $G$ by contracting component $C_1$ to a single point $u_1$. Of course $G'$ is planar since $G$ is and $G'$ has a single point $u_1$ of degree 5 and all others of degree 3. It is possible that by contracting $C_1$ to a point we have introduced parallel lines in $G'$. However, if $p' = |V(G')|$, $q' = |E(G')|$ and $r'$ denotes the number of faces in any imbedding of graph $G'$ in the plane, using Euler's formula, we have $\sum_i (6-i)f'_i = 6r' - 2q' = 6r' - 3p' - 2 = 6(2+q'-p') - 3p' - 2 = 16$, since in $G'$ we have $2q' = 3p' + 2$. So, in particular, in $G'$ we have $4f'_2 + 3f'_3 + 2f'_4 + f'_5 \geq 16$.

Now since induced subgraph $G[S]$ contains only 2 lines, there can be at most 4 odd faces in $G'$. Hence $f'_3 + f'_5 \leq 4$. Thus $2f'_2 + f'_3 + f'_4 \geq 6$. But $G$ has no faces of size 3 or 4, so all triangular or quadrilateral faces in $G'$ must contain $u_1$ in their boundary. Thus in $G'$ we also have But since $\deg u_1 = 5$ in $G'$, we also have $f'_3 + f'_4 \leq 5$. Hence $f'_2 \geq 1$. This implies that in $G$ we have $|V(L) \cap S| \leq 4$ where $L$ denotes the set of lines joining component $C_1$ to $G[V(G) - V(C_1)]$. There are thus only 2 possible values for $|V(L) \cap S|$ and we now proceed to treat each.

**Case 3.1.** Suppose $|V(L) \cap S| = 3$.

Let $v_1$, $v_2$ and $v_3$ be the 3 points of $S$ adjacent to points of $C_1$. Note that if any of these $v_i$'s is adjacent to 3 points of $C_1$, then the other two together form a cutset of $G$ of size 2, contradicting the hypothesis that $G$ is 3-connected. So we must have 2 of the $v_i$'s incident with 2 lines to $C_1$ and the third $v_i$ incident with 1 line to $C_1$. Without loss of generality, assume that $v_1, v_2$ are each adjacent to 2 points of $C_1$ and $v_3$ is adjacent to 1 point of $C_1$.

Note that since $G$ is 3-connected, $\{v_1, v_2, v_3\}$ is an independent set. Now let $f_i$ be the line joining $v_i$ to a point not in $C_1$ for $i = 1, 2$, and let $f_3$ be the line joining $v_3$ to $C_1$. Then $\{f_1, f_2, f_3\}$ is a cutset in $G$ separating $G_2 = G[V(C_1) \cup \{v_1, v_2\}]$ from $G_3 = G[V(G) - V(C_1) - \{v_1, v_2\}]$. (See Figure 3.3.) Moreover, $G_2$ contains a cycle since it contains component $C_1$.

We claim that $G_3$ also contains a cycle. Suppose not. Then $G_3$ is a forest containing the 2 lines $e_1$ and $e_2$ and hence is a forest containing at least 2 endpoints. Thus $G_3$ sends at least 4 lines to $G_2$, a contradiction.

Thus $\{f_1, f_2, f_3\}$ is a *cyclic* cutset of size 3 in $G$, contradicting the hypothesis that $G$ is cyclically 4-connected.

**Case 3.2.** Suppose $|V(L) \cap S| = 4$.

Let $v_1$ be the point of $S$ adjacent to 2 points of $C_1$ and $v_2, v_3, v_4$ be the rest of $V(L) \cap S$. Let $w$ be the point adjacent to $v_1$ which lies outside of $V(C_1)$.

**FIGURE 3.3.**

First suppose $v_1$ is adjacent to one of $v_2, v_3$ or $v_4$; say, without loss of generality, to $v_2$ via line $e_1$. Let the lines joining $v_i$ to $C_1$ be $f_i$ for $i = 2, 3, 4$. Finally, let $g$ be the line incident with $v_2$, where $g \neq e_1$ or $f_2$. Then $\{g, f_3, f_4\}$ is a cutset in $G$ separating $J_1 = G[V(C_1) \cup \{v_1, v_2\}]$ from the rest of $G$. Let us denote $G[V(G) - V(J_1)]$ by $J_2$. Note first that $J_1$ contains a cycle since $C_1$ does.

**Claim 5.** Subgraph $J_2$ contains a cycle.

Suppose not. We know that $J_2$ contains line $e_2$ and hence is a forest with at least 2 endpoints. So $J_2$ sends at least 4 lines to $J_1$, a contradiction.

Thus $\{g, f_3, f_4\}$ is a *cyclic* cutset in $G$, contradicting the hypothesis that $G$ is cyclically 4-connected.

So we may assume that $v_1$ is adjacent to none of the points $v_2, v_3, v_4$; that is, $w \notin \{v_2, v_3, v_4\}$.

Now contract the subgraph $G[V(C_1) \cup \{v_1\}]$ to a single point $c_1$ and

FIGURE 3.4.

call the resulting graph $G'''$. (See Figure 3.4.)

The graph $G'''$ has all points of degree 3, with the single exception of point $c_1$ which has degree 4. Let $p'''$, $q'''$ and $r'''$ denote the number of points, lines and faces of graph $G'''$ respectively. Then doing an Euler face count in $G'''$, we have $\sum_i (6-i) f_i''' = 6r''' - 2q''' = 6r''' - (3p'''+1) = 14$. So in particular, we have $4f_2''' + 3f_3''' + 2f_4''' + f_5''' \geq 14$.

But $w \notin \{v_2, v_3, v_4\}$, so $f_2''' = 0$, and so in $G'''$ we have:

$$3f_3''' + 2f_4''' + f_5''' \geq 14 \tag{$A$}$$

We also know that since $\deg c_1 = 4$ in $G'''$,

$$f_3''' + f_4''' \leq 4. \tag{$B$}$$

Now either $w \in S$ or $w$ is a singleton component of $G - S$ different from $C_1$.

Suppose that $w \in S$. Then any triangle or quadrilateral in $G'''$ must use one of the lines $e_1$ or $e_2$, so

$$f_3''' + f_5''' \leq 4. \tag{$C$}$$

But then if we compute $(A) - ((B) + (C))$ we obtain $f_3''' + f_4''' \geq 6$ which contradicts inequality (B).

So we may suppose that $w$ is a singleton component of $G - S$.

Let line $c_1 w$ be denoted by $h$ in $G'''$. Let the 4 faces at point $c_1$ be denoted $F_1, F_2, F_3$ and $F_4$ as shown in Figure 3.5.

**FIGURE 3.5.**

**FIGURE 3.6.**

Note that since lines $e_1$ and $e_2$ are independent, at most one of $F_2$ and $F_3$ is a triangle.

**3.2.1.** First suppose that $F_4$ is a triangle.

**3.2.1.1.** Suppose that face $F_1$ is also a triangle. (See Figure 3.6.) Then if $F_3$ is a triangle, the points $\{v_2, v_3\}$ are a cutset in $G$, a contradiction. Thus $F_3$ is not a triangle and by symmetry, neither is $F_2$. But then let $h_2 = v_2x$, $x \notin \{c_1, w\}$, $h_3 = v_3c_1$ and $h_4 = v_4y$, $y \notin \{c_1, w\}$. Then $\{h_2, h_3, h_4\}$ is a set of 3 independent lines separating cycle $v_2c_1wv_2$, for

**FIGURE 3.7.**

example, from a subgraph $H'''$ of $G'''$ containing 3 points of degree 2 in $H'''$, namely $x, v_3$, and $y$. Moreover, since $G$ is 3-connected, subgraph $H'''$ must be connected and hence $\{h_2, h_3, h_4\}$ is a *cyclic* 3-cut, a contradiction.

**3.2.1.2.** So suppose face $F_1$ is not a triangle. Since $e_1$ and $e_2$ are independent, at most one of $F_2$ and $F_3$ is a triangle. Hence $f_3''' \leq 2$.

If $f_3''' = 1$, then $f_4''' \leq 3$ and $f_5''' \leq 4$. But then $2f_3''' + f_4''' + f_5''' \leq 9$ and combining this inequality with inequality (A), we obtain $f_3''' + f_4''' \geq 5$ which contradicts inequality (B).

So we may conclude that $f_3''' = 2$ and hence exactly one of $F_2$ and $F_3$ is a triangle.

**3.2.1.2.1.** Suppose $F_2$ is a triangle. In particular, suppose line $e_1$ joins points $v_2$ and $v_3$. Then $e_2$ is not incident with $v_2$, so $v_2$ must be adjacent to a point $x$ in $G - S$, where $x \neq w$. But $x$ cannot be adjacent to $w$, so face $F_1$ cannot be a quadrilateral. So $f_4''' \leq 1$. It then follows that $f_3''' + f_4''' + f_5''' \leq 6$ and hence $f_3''' + f_4''' \geq 6$, contradicting inequality (B).

**3.2.1.2.2.** Now suppose $F_3$ is a triangle. (And face $F_2$ is *not* a triangle.) So we may assume that line $e_1$ joins points $v_3$ and $v_4$.

We know that $F_1$ is not a triangle.

If face $F_1$ is not a quadrilateral, we get the same contradiction that we obtained in Subcase 3.2.1.2.1, so suppose $F_1$ is a quadrilateral. (See Figure 3.7.)

Thus line $e_2$ must join point $v_2$ to a point $y$ in $S$ and, in addition,

**FIGURE 3.8.**

$y$ is adjacent to $w$.

So $f_3''' = 2$, $f_4''' \leq 2$ and $f_5''' \leq 2$. But then $2f_3''' + f_4''' + f_5''' \leq 8$ and combining this inequality with inequality (A), we obtain $f_3''' + f_4''' \geq 6$ which contradicts inequality (B).

**3.2.2.** Suppose now that $F_4$ is not a triangle. (And by symmetry, we may also suppose that $F_1$ is not a triangle either.) Thus $f_3''' \leq 1$. Now if $f_3''' = 1$, then $f_4''' \leq 3$ and $f_5''' \leq 3$, while if $f_3''' = 0$, then $f_4''' \leq 4$ and $f_5''' \leq 4$. But in either case, $2f_3''' + f_4''' + f_5''' \leq 8$ and once again combining this inequality with inequality (A), it follows that $f_3''' + f_4''' \geq 6$ and again we have a contradiction of inequality (B).

This completes the proof of the theorem. ■

The following corollary is now immediate.

**3.2. COROLLARY.** *If $G$ is a cubic, 3-connected, planar graph and, in addition, is cyclically 5-connected, then $G$ is 2-extendable.* ■

We conclude with several remarks as to the sharpness of Theorem 3.1. First we note that there are graphs which satisfy the hypotheses of Theorem 3.1, but not those of Corollary 3.2. Such a graph is displayed in Figure 3.8.

We now observe that our theorem is best possible in the sense that we cannot weaken the assumption that the cyclic connectivity is 4 to the assumption that it is only 3 in Theorem 3.1. The graph in Figure 3.9 is cubic, 3-connected and planar without any triangles or quadrilaterals, but it is *not* 2-extendable. (Lines $e_1$ and $e_2$ do not extend to a p.m.)

FIGURE 3.9.

FIGURE 3.10.

Finally, in Figure 3.10 we exhibit a graph which is cubic, 3-connected, planar and cyclically 4-connected, but not 2-extendable. (Lines $e_1$ and $e_2$ do not extend.) Of course, by Theorem 3.1, such a graph must contain a quadrilateral and the example we display contains two such.

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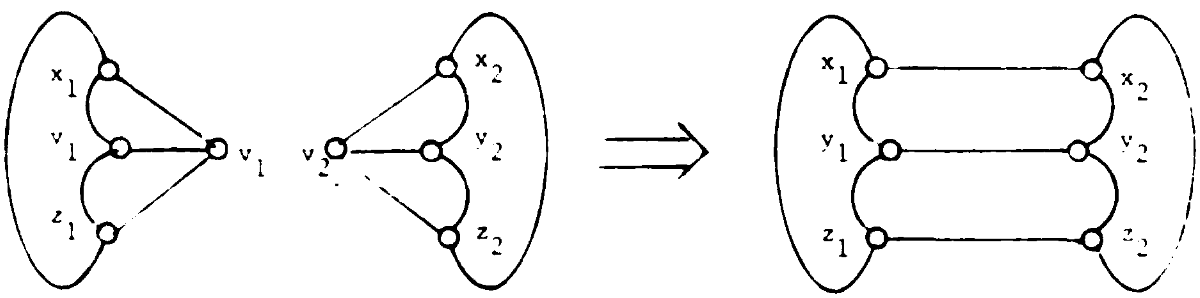


Figure 2.1

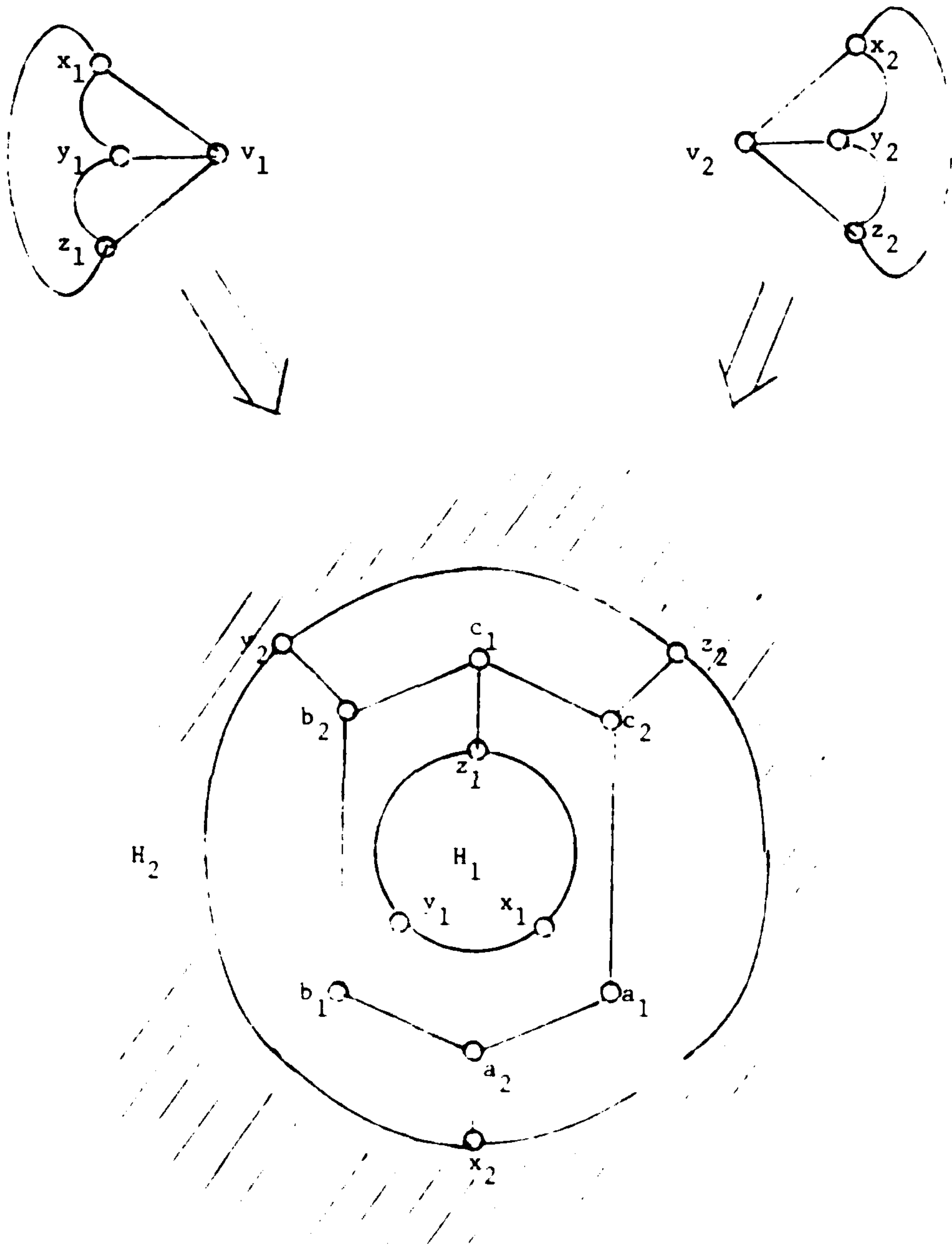


Figure 2.2

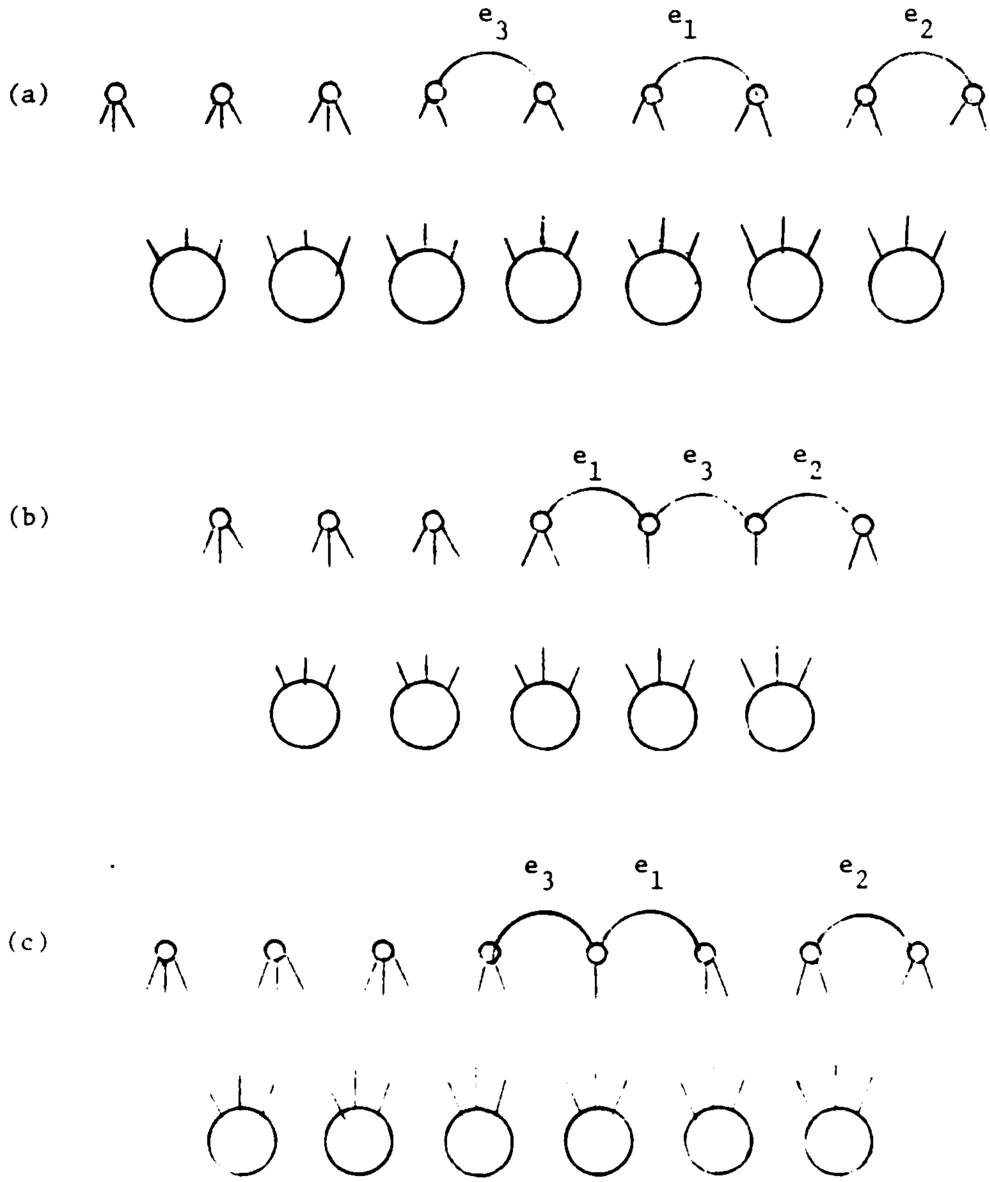


Figure 3.1

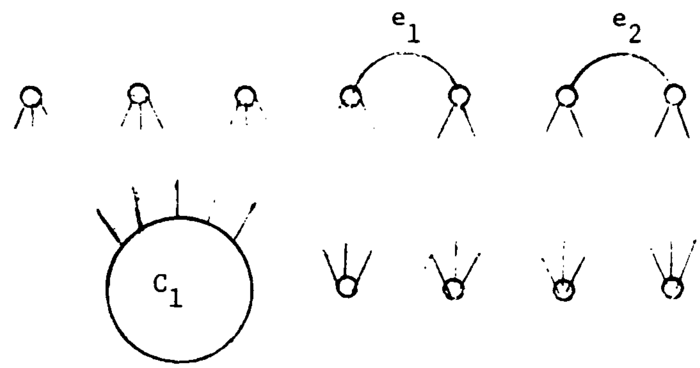


Figure 3.2

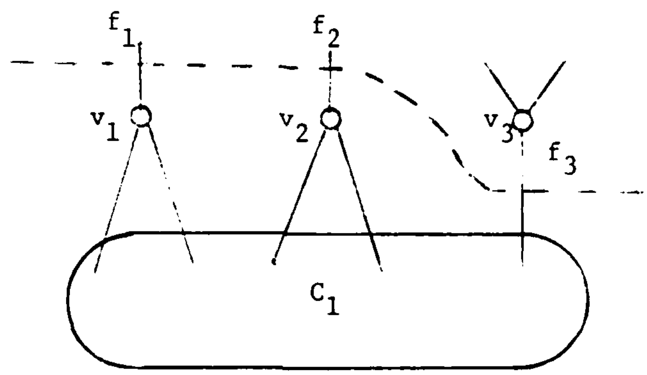


Figure 3.3

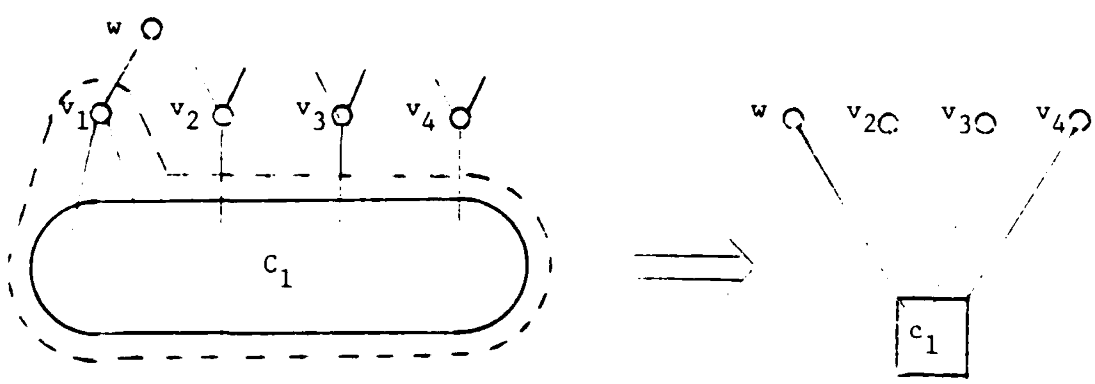


Figure 3.4

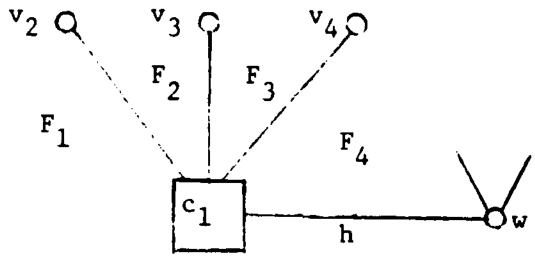


Figure 3.5

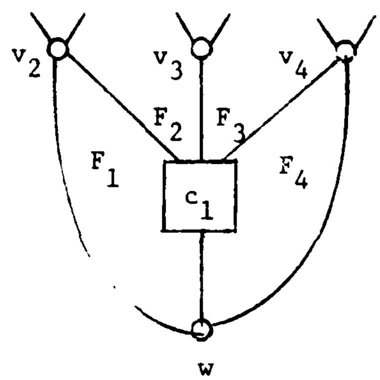


Figure 3.6

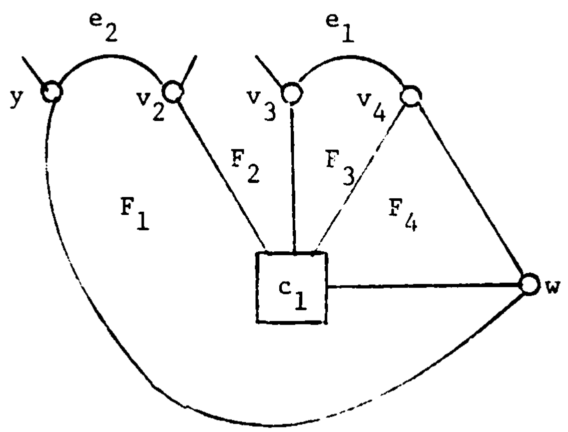


Figure 3.7

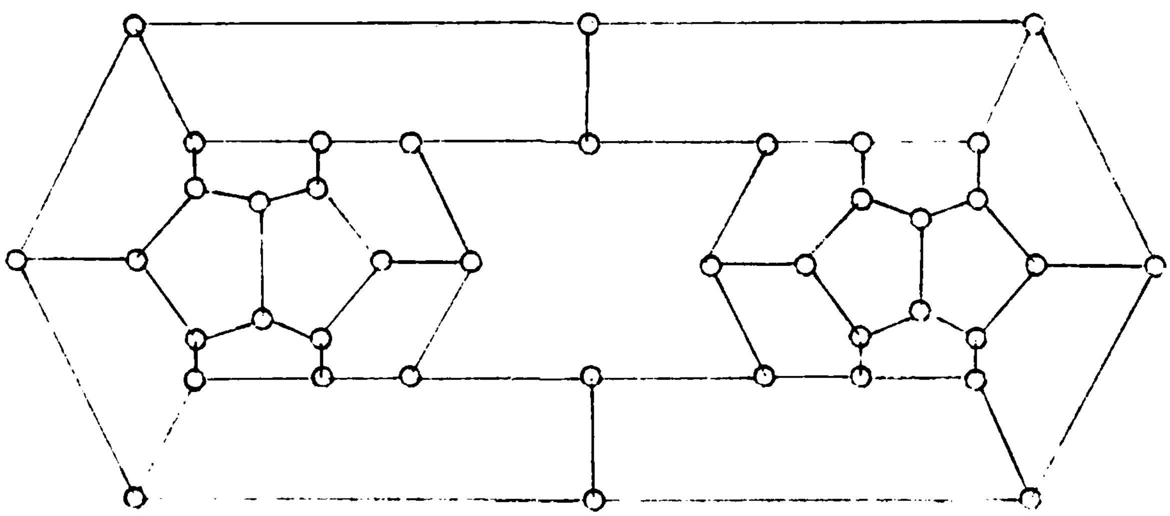

Figure 3.8

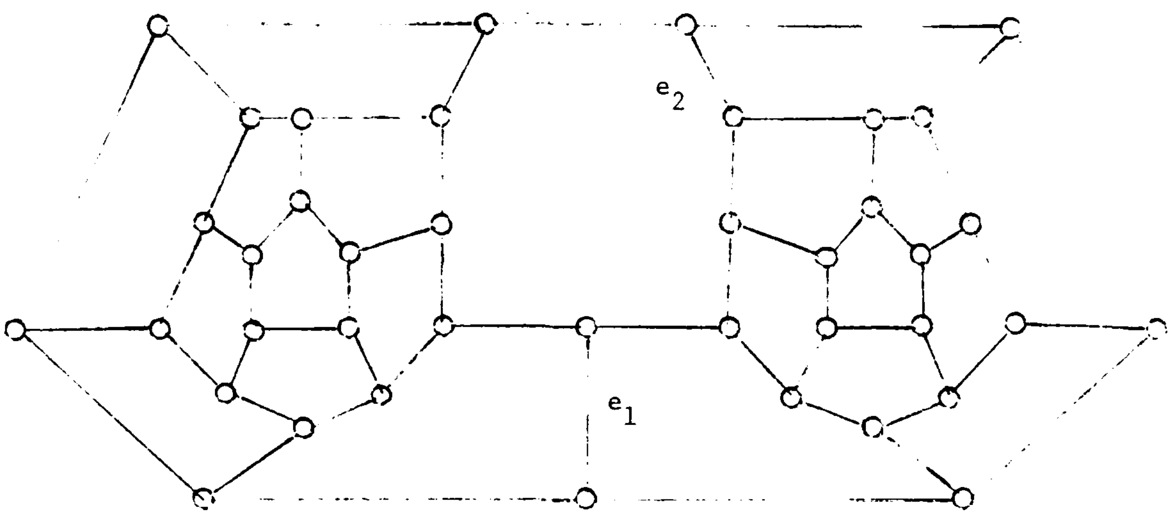


Figure 3.9

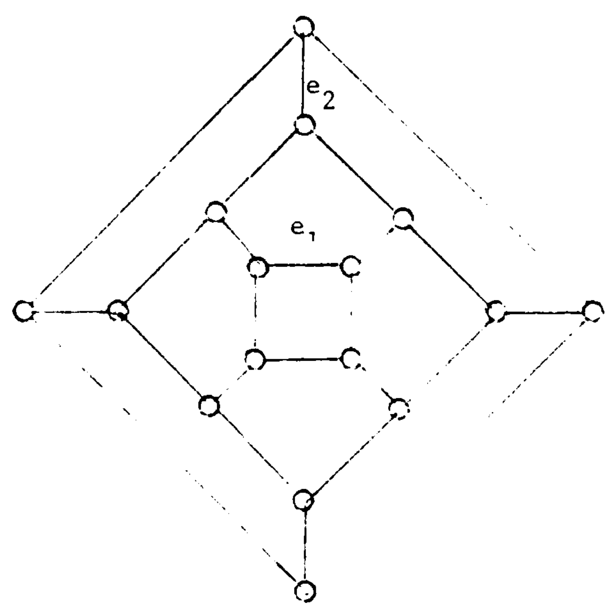


Figure 3.10

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